On the generalized Feynman-Kac transformation for nearly symmetric Markov processes

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Abstract

Suppose X is a right process which is associated with a non-symmetric Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(E;m)$. For $u \in D(\mathcal{E})$, we have Fukushima's decomposition: $\tilde{u}(X_t) - \tilde{u}(X_0) = M_t^u + N_t^u$. In this paper, we investigate the strong continuity of the generalized Feynman-Kac semigroup defined by $P_t^u f(x) = E_x[e^{N_t^u} f(X_t)]$. Let $Q^u(f,g) = \mathcal{E}(f,g) + \mathcal{E}(u, fg)$ for $f, g \in D(\mathcal{E})_b$. Denote by J_1 the dissymmetric part of the jumping measure J of $(\mathcal{E}, D(\mathcal{E}))$. Under the assumption that J_1 is finite, we show that $(Q^u, D(\mathcal{E})_b)$ is lower semi-bounded if and only if there exists a constant $\alpha_0 \geq 0$ such that $\|P_t^u\|_2 \leq e^{\alpha_0 t}$ for every $t > 0$. If one of these conditions holds, then $(P_t^u)_{t\geq 0}$ is strongly continuous on $L^2(E;m)$. If X is equipped with a differential structure, then this result also holds without assuming that J_1 is finite.

Keywords: Non-symmetric Dirichlet form; generalized Feynman-Kac semigroup; strong continuity; lower semi-bounded; Beurling-Deny formula; LeJan's transformation rule

1 Introduction

Let E be a metrizable Lusin space and $X = ((X_t)_{t\geq 0}, (P_x)_{x\in E})$ be a right (continuous strong Markov) process on E (cf. [22, IV, Definition 1.8]). Suppose that X is associated with a (non-symmetric) Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(E;m)$, where m is a σ-finite measure on the Borel σ-algebra $\mathcal{B}(E)$ of E. Then, by [22, IV, Theorem 6.7] (cf. also [15, Theorem 3.22]), $(\mathcal{E}, D(\mathcal{E}))$ is quasi-regular. Moreover, $(\mathcal{E}, D(\mathcal{E}))$ is quasi-homeomorphic to a regular Dirichlet form (see [11]). We refer the reader to [22] and [17] for the theory of Dirichlet forms. The notations and terminologies of this paper follow [22] and [17].

Let $u \in D(\mathcal{E})$. Then, we have Fukushima's decomposition (cf. [22, VI, Theorem 2.5])

$$\tilde{u}(X_t) - \tilde{u}(X_0) = M_t^u + N_t^u,$$

where $\tilde{u}$ is a quasi-continuous m-version of u, M_t^u is a square integrable martingale additive functional (MAF) and N_t^u is a continuous additive functional (CAF) of zero energy. For $x \in E$, denote by E_x the expectation with respect to (w.r.t.) P_x. Define the generalized Feynman-Kac transformation

$$P_t^u f(x) = E_x[e^{N_t^u} f(X_t)], \quad f \geq 0 \text{ and } t \geq 0.$$

In this paper, we will investigate the strong continuity of the semigroup $(P_t^u)_{t\geq 0}$ on $L^2(E;m)$.

The strong continuity of generalized Feynman-Kac semigroups for symmetric Markov processes has been studied extensively by many people. Note that in general $(N_t^u)_{t\geq 0}$ is not of finite variation (cf. [17, Example 5.5.2]). Hence the classical results of Albeverio and Ma given in [1] do not apply directly. Under the assumption that X is the standard d-dimensional Brownian motion, u is a bounded continuous function on $\mathbf{R}^d$ and $|\nabla u|^2$ belongs to the Kato class, Glover et al. proved in [18] that $(P_t^u)_{t\geq 0}$ is a strongly continuous semigroup on $L^2(\mathbf{R}^d; dx)$. Moreover, they gave an explicit representation for the closed quadratic form corresponding to $(P_t^u)_{t\geq 0}$. In [25], Zhang generalized the results of [18] to symmetric Lévy processes on $\mathbf{R}^d$ and removed the assumption that u is bounded continuous. Furthermore, Z.Q. Chen and Zhang established in [13] the corresponding results for general symmetric Markov processes via Girsanov transformation. They proved that if $\mu_{\langle u\rangle}$, the energy measure of u, is a measure of the Kato class, then $(P_t^u)_{t\geq 0}$ is a strongly continuous semigroup on $L^2(E;m)$. Also, they characterized the closed quadratic form corresponding to $(P_t^u)_{t\geq 0}$. In [16], Fitzsimmons and Kuwae established the strong continuity of $(P_t^u)_{t\geq 0}$ under the assumption that X is a symmetric diffusion process and $\mu_{\langle u\rangle}$ is a measure of the Hardy class. Furthermore, Z.Q. Chen et al. established in [9] the strong continuity of $(P_t^u)_{t\geq 0}$ for general symmetric Markov processes under the assumption that $\mu_{\langle u\rangle}$ is a measure of the Hardy class.

All the results mentioned above give sufficient conditions for $(P_t^u)_{t\geq 0}$ to be strongly continuous, where $\mu_{\langle u\rangle}$ is assumed to be of either the Kato class or the Hardy class. In [5], under the assumption that X is a symmetric diffusion process, C.Z. Chen and Sun showed that the semigroup $(P_t^u)_{t\geq 0}$ is strongly continuous on $L^2(E;m)$ if and only if the bilinear form $(Q^u, D(\mathcal{E})_b)$ is lower semi-bounded. Here and henceforth

$$Q^u(f,g) := \mathcal{E}(f,g) + \mathcal{E}(u, fg), \quad f, g \in D(\mathcal{E})_b := D(\mathcal{E}) \cap L^\infty(E;m). \tag{1.1}$$

Furthermore, C.Z. Chen et al. generalized this result to general symmetric Markov processes in [4]. In [10], Z.Q. Chen et al. studied general perturbations of symmetric Markov processes and gave another proof for the equivalence of the strong continuity of $(P_t^u)_{t\geq 0}$ and the lower semi-boundedness of $(Q^u, D(\mathcal{E})_b)$.

The aim of this paper is to study the strong continuity problem of generalized Feynman-Kac semigroups for nearly symmetric Markov processes. Note that many useful tools of symmetric Dirichlet forms, e.g. time reversal and Lyons-Zheng decomposition, do not apply well to the non-symmetric setting. That makes the

problem more difficult. Also, we would like to point out that the Girsanov transformed process of X induced by M_t^u and the Girsanov transformed process of $\hat{X}$ induced by $\hat{M}_t^u$ are not in duality in general (cf. [6]), where $\hat{X}$ is the dual process of X and $\hat{M}_t^u$ is the martingale part of $\tilde{u}(\hat{X}_t)-\tilde{u}(\hat{X}_0)$. The method of this paper is inspired by [4] and [10]. We will combine the h-transform method of [4] and the localization method used in [10]. It is worth to point out that the Beurling-Deny formula given in [20] and LeJan's transformation rule developed in [21] play a crucial role in this paper.

Denote by J and K the jumping and killing measures of $(\mathcal{E},D(\mathcal{E}))$, respectively. Write $\hat{J}(dx,dy)=J(dy,dx)$. Denote by $J_1:=(J-\hat{J})^+$ the positive part of the Jordan decomposition of $J-\hat{J}$. J_1 is called the dissymmetric part of J. Note that $J_0:=J-J_1$ is the largest symmetric σ-finite positive measure dominated by J. Denote by d the diagonal of the product space $E\times E$; and denote by $\|\cdot\|_2$ and $(\cdot,\cdot)_m$ the norm and inner product of $L^2(E;m)$, respectively.

Now we can state the main results of the paper.

Theorem 1.1. *Suppose that X is a right process which is associated with a (non-symmetric) Dirichlet form $(\mathcal{E},D(\mathcal{E}))$ on $L^2(E;m)$. Let $u\in D(\mathcal{E})$. Assume that $J_1(E\times E\backslash d)<\infty$. Then the following two conditions are equivalent:*
(i) There exists a constant $\alpha_0\geq 0$ such that

$$Q^u(f,f)\geq -\alpha_0(f,f)_m,\quad \forall f\in D(\mathcal{E})_b.$$

(ii) There exists a constant $\alpha_0\geq 0$ such that

$$\|P_t^u\|_2\leq e^{\alpha_0 t},\quad \forall t>0.$$

Furthermore, if one of these conditions holds, then the semigroup $(P_t^u)_{t\geq 0}$ is strongly continuous on $L^2(E;m)$.

Theorem 1.2. *Let U be an open set of $\mathbf{R^d}$ and m be a positive Radon measure on U with* supp$[m]=U$*. Suppose that X is a right process which is associated with a (non-symmetric) Dirichlet form $(\mathcal{E},D(\mathcal{E}))$ on $L^2(U;m)$ such that $C_0^\infty(U)$ is dense in $D(\mathcal{E})$. Then the conclusions of Theorem 1.1 remain valid without assuming that $J_1(E\times E\backslash d)<\infty$.*

The rest of this paper is organized as follows. In Section 2, we give the proof of Theorems 1.1 and 1.2. In Section 3, we give two examples, one is finite-dimensional and the other one is infinite-dimensional.

2 Proof of the Main Results

In this section, we will prove Theorems 1.1 and 1.2. By quasi-homeomorphism, we assume without loss of generality that X is a Hunt process and $(\mathcal{E},D(\mathcal{E}))$ is a

regular (non-symmetric) Dirichlet form on $L^2(E;m)$, where E is a locally compact separable metric space and m is a positive Radon measure on E with $\mathrm{supp}[m]=E$. We denote by Δ and ζ the cemetery and lifetime of X, respectively. It is known that every $f\in D(\mathcal{E})$ has a quasi-continuous m-version. To simplify notation, we still denote this version by f.

Let $u\in D(\mathcal{E})$. By [22, III, Proposition 1.5], there exists $|u|_E\in D(\mathcal{E})$ such that $|u|_E\geq |u|\ m-a.e.$ on E and $\mathcal{E}_1(|u|_E,w)\geq 0$ for all $w\in D(\mathcal{E})$ with $w\geq 0\ m-a.e.$ on E. Similar to [17, Theorems 2.2.1 and 2.2.2], we can show that there exists a positive Radon measure η_u on E such that

$$\mathcal{E}_1(|u|_E,w)=\int_E wd\eta_u,\quad w\in D(\mathcal{E}). \tag{2.1}$$

Define

$$u^*:=u+|u|_E. \tag{2.2}$$

Then, u^* has a quasi-continuous m-version which is nonnegative q.e. on E. Moreover, there exists an $\mathcal{E}$-nest $\{F_n\}_{n\in\mathbf{N}}$ consisting of compact sets of E such that u^* is continuous and hence bounded on F_n for each $n\in\mathbf{N}$. Define $\tau_{F_n}=\inf\{t>0|X_t\in F_n^c\}$. By [22, IV, Proposition 5.30], $\lim_{n\to\infty}\tau_{F_n}=\zeta$ P_x-a.s. for q.e. $x\in E$.

Let (N,H) be a Lévy system of X and ν be the Revuz measure of H. Define

$$B_t=\sum_{s\leq t}\left[e^{(u^*(X_{s-})-u^*(X_s))}-1-(u^*(X_{s-})-u^*(X_s))\right]. \tag{2.3}$$

Note that for any $M>0$ there exists $C_M>0$ such that $(e^x-1-x)\leq C_Mx^2$ for all x satisfying $x\leq M$. Since $(u^*(X_{t-}))_{t\geq 0}$ is locally bounded, $(u^*(X_t))_{t\geq 0}$ is nonnegative and M^{-u^*} is a square integrable martingale for q.e. $x\in E$, hence $(B_t)_{t\geq 0}$ is locally integrable on $[0,\zeta)$ for q.e. $x\in E$. Here and henceforth the phrase "on $[0,\zeta)$" is understood as "on the optional set $[[0,\zeta))$ of interval type" in the sense of [19, Chap. VIII, 3]. By [17, (A.3.23)], one finds that the dual predictable projection of $(B_t)_{t\geq 0}$ is given by

$$B_t^p=\int_0^t\int_{E_\triangle}[e^{(u^*(X_s)-u^*(y))}-1-(u^*(X_s)-u^*(y))]N(X_s,dy)dH_s.$$

We set

$$M_t^d=B_t-B_t^p \tag{2.4}$$

and denote

$$M_t=M_t^{-u^*}+M_t^d. \tag{2.5}$$

Note that for any $M>0$ there exists $D_M>0$ such that $(e^x-1-x)^2\leq D_Mx^2$ for all x satisfying $x\leq M$. Since $(u^*(X_{t-}))_{t\geq 0}$ is locally bounded, $(u^*(X_t))_{t\geq 0}$ is nonnegative and M^{-u^*} is a square integrable martingale for q.e. $x\in E$, hence $(M_t^d)_{t\geq 0}$ is a locally square integrable MAF on $[0,\zeta)$ for q.e. $x\in E$ by [19, Theorem

7.40]. Therefore $(M_t)_{t\geq 0}$ is a locally square integrable MAF on $[0,\zeta)$ for q.e. $x\in E$. We denote the Revuz measure of $(<M>_t)_{t\geq 0}$ by $\mu_{<M>}$ (cf. [8, Remark 2.2]).

Let $M_t^{-u^*,c}$ be the continuous part of $M_t^{-u^*}$. Define

$$A_t^{-u^*}=B_t^p+\frac{1}{2}<M^{-u^*,c}>_t\ . \tag{2.6}$$

Then $(A_t^{-u^*})_{t\geq 0}$ is a positive CAF (PCAF). Denote by μ_{-u^*} the Revuz measure of $(A_t^{-u^*})_{t\geq 0}$. Then

$$\begin{aligned}\mu_{-u^*}(dx) &= \int_{E_\triangle}[e^{(u^*(x)-u^*(y))}-1-(u^*(x)-u^*(y))]N(x,dy)\nu(dx)\\ &\quad+\frac{1}{2}\mu_{<M^{-u^*,c}>}(dx).\end{aligned}\tag{2.7}$$

Define

$$\mu_{-u}:=\mu_{-u^*}+\eta_u-|u|_E m \tag{2.8}$$

and

$$\mu'_{-u}:=\mu_{-u^*}+\eta_u+|u|_E m.$$

Recall that a smooth measure μ is said to be of the Kato class if

$$\lim_{t\to 0}\inf_{\mathrm{Cap}(N)=0}\sup_{x\in E-N}E_x[A_t^\mu]=0,$$

where $(A_t^\mu)_{t\geq 0}$ is the PCAF associated with μ. Denote by S_K the Kato class of smooth measures. Similar to [2, Theorem 2.4], we can show that there exists an $\mathcal{E}$-nest $\{F'_n\}_{n\in\mathbf{N}}$ consisting of compact sets of E such that $I_{F'_n}(\mu_{<M>}+\mu'_{-u})\in S_K$. To simplify notation, we still use F_n to denote $F_n\cap F'_n$ for $n\in\mathbf{N}$. Let E_n be the fine interior of F_n w.r.t. X. Define $D(\mathcal{E})_n:=\{f\in D(\mathcal{E})|f=0 \text{ q.e. on } E_n^c\}$, $\tau_{E_n}=\inf\{t>0|X_t\in E_n^c\}$ and

$$\bar{P}_t^{u,n}f(x):=E_x[e^{M_t^{-u^*}-N_t^{|u|_E}}f(X_t);t<\tau_{E_n}].$$

2.1 The bilinear form associated with $(\bar{P}_t^{u,n})_{t\geq 0}$ on $L^2(E_n;m)$

For $n\in\mathbf{N}$, we define the bilinear form $(\bar{Q}^{u,n},D(\mathcal{E})_n)$ by

$$\bar{Q}^{u,n}(f,g)=\mathcal{E}(f,g)-\int_E gd\mu_{<M^f,M>}-\int_E fgd\mu_{-u},\quad f,g\in D(\mathcal{E})_n. \tag{2.9}$$

By [12, Lemma 4.3], for every $\varepsilon>0$, there exists a constant $A_\varepsilon^n>0$ such that

$$\int_E w^2d(\mu_{<M>}+\mu'_{-u})\leq\varepsilon\mathcal{E}(w,w)+A_\varepsilon^n\|w\|_2^2,\quad w\in D(\mathcal{E})_n.$$

Suppose that $|\mathcal{E}(f,g)|\le k_1\mathcal{E}_1(f,f)^{\frac12}\mathcal{E}_1(g,g)^{\frac12}$ for all $f,g\in D(\mathcal{E})$ and some constant $k_1>0$. Then

$$\begin{aligned}
|\bar{Q}^{u,n}(f,g)| &\le k_1\mathcal{E}_1(f,f)^{\frac12}\mathcal{E}_1(g,g)^{\frac12}+\left(\int_E d\mu_{<M^f>}\right)^{\frac12}\left(\int_E g^2 d\mu_{<M>}\right)^{\frac12}\\
&\quad+\left(\int_E f^2 d\mu'_{-u}\right)^{\frac12}\left(\int_E g^2 d\mu'_{-u}\right)^{\frac12}\\
&\le k_1\mathcal{E}_1(f,f)^{\frac12}\mathcal{E}_1(g,g)^{\frac12}+(\max(\varepsilon,A^n_\varepsilon))^{\frac12}[2\mathcal{E}(f,f)]^{\frac12}\mathcal{E}_1(g,g)^{\frac12}\\
&\quad+\max(\varepsilon,A^n_\varepsilon)\cdot\mathcal{E}_1(f,f)^{\frac12}\mathcal{E}_1(g,g)^{\frac12}\\
&\le \theta_n\mathcal{E}_1(f,f)^{\frac12}\mathcal{E}_1(g,g)^{\frac12}, \qquad (2.10)
\end{aligned}$$

where $\theta_n:=(k_1+\sqrt{2\max(\varepsilon,A^n_\varepsilon)}+\max(\varepsilon,A^n_\varepsilon))$.

Fix an $\varepsilon<(\sqrt{2}-1)/(\sqrt{2}+1)$ and set $\alpha_n:=2A^n_\varepsilon$. Then

$$\begin{aligned}
\bar{Q}^{u,n}_{\alpha_n}(f,f) &:= \bar{Q}^{u,n}(f,f)+\alpha_n(f,f)\\
&\ge \mathcal{E}(f,f)-\left(\int_E d\mu_{<M^f>}\right)^{\frac12}\left(\int_E f^2 d\mu_{<M>}\right)^{\frac12}\\
&\quad-\int_E f^2 d\mu'_{-u}+\alpha_n(f,f)\\
&\ge \mathcal{E}(f,f)-(\varepsilon\ \mathcal{E}(f,f)+A^n_\varepsilon\|f\|_2^2)^{\frac12}[2\mathcal{E}(f,f)]^{\frac12}\\
&\qquad-(\varepsilon\ \mathcal{E}(f,f)+A^n_\varepsilon\|f\|_2^2)+\alpha_n(f,f)\\
&\ge \mathcal{E}(f,f)-\frac{1}{\sqrt{2}}((1+\varepsilon)\mathcal{E}(f,f)+A^n_\varepsilon\|f\|_2^2)\\
&\qquad-(\varepsilon\ \mathcal{E}(f,f)+A^n_\varepsilon\|f\|_2^2)+\alpha_n(f,f)\\
&\ge \frac{\sqrt{2}-1-(\sqrt{2}+1)\varepsilon}{\sqrt{2}}\mathcal{E}(f,f)+\frac{(\sqrt{2}-1)A^n_\varepsilon}{\sqrt{2}}\|f\|_2^2. \qquad (2.11)
\end{aligned}$$

By (2.10), (2.11) and [22, I, Proposition 3.5], we know that $(\bar{Q}^{u,n}_{\alpha_n},D(\mathcal{E}))$ is a coercive closed form on $L^2(E_n;m)$.

Theorem 2.1. *For each $n\in\mathbf{N}$, $(\bar{P}^{u,n}_t)_{t\ge0}$ is a strongly continuous semigroup of bounded operators on $L^2(E_n;m)$ with $\|\bar{P}^{u,n}_t\|_2\le e^{\beta_n t}$ for every $t>0$ and some constant $\beta_n>0$. Moreover, the coercive closed form associated with $(e^{-\beta_n t}\bar{P}^{u,n}_t)_{t\ge0}$ is given by $(\bar{Q}^{u,n}_{\beta_n},D(\mathcal{E})_n)$.*

Proof. The proof is much similar to that of [16, Theorem 1.1], which is based on a key lemma (see [16, Lemma 3.2]) and a remarkable localization method. In fact, the proof of our Theorem 2.1 is simpler since $I_{F_n}(\mu_{<M>}+\mu'_{-u})$ is of the Kato class instead of the Hardy class and there is no time reversal part in the semigroup $(\bar{P}^{u,n}_t)_{t\ge0}$. We omit the details of the proof here and only give the following key lemma, which is the counterpart of [16, Lemma 3.2]. □

Lemma 2.2. *Let $(L^{\bar{Q}^{u,n}}, D(L^{\bar{Q}^{u.n}}))$ be the generator of $(\bar{Q}^{u,n}, D(\mathcal{E})_n)$. Then, for any $f \in D(L^{\bar{Q}^{u,n}})$, we have*

$$\begin{aligned} f(X_t)e^{M_t^{-u^*}-N_t^{|u|_E}} &= f(X_0)+\int_0^t e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}}dM_s^f \\ &\quad +\int_0^t e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}}f(X_{s-})dM_s \\ &\quad +\int_0^t e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}}L^{\bar{Q}^{u,n}}f(X_s)ds \end{aligned} \tag{2.12}$$

P_m-a.s. on $\{t<\tau_{E_n}\}$.

Proof. Let $f \in D(L^{\bar{Q}^{u,n}})$ and $g \in D(\mathcal{E})_n$. Then, by (2.9), we get

$$\begin{aligned} \mathcal{E}(f,g) &= \bar{Q}^{u,n}(f,g)+\int_E g d\mu_{<M^f,M>}+\int_E fg d\mu_{-u} \\ &= -(L^{\bar{Q}^{u,n}}f,g)+\int_E g d\mu_{<M^f,M>}+\int_E fg d\mu_{-u}. \end{aligned} \tag{2.13}$$

By (2.1), (2.13) and [23, Theorem 5.2.7], we find that $(N_t^{|u|_E})_{t\geq 0}$ is a CAF of bounded variation and

$$N_t^f = \int_0^t L^{\bar{Q}^{u,n}}f(X_s)ds - <M^f,M>_t - \int_0^t f(X_s)d(A_s^{-u^*}-N_s^{|u|_E})$$

for $t<\tau_{E_n}$. Therefore, for $t<\tau_{E_n}$, we have

$$\begin{aligned} f(X_t)-f(X_0) &= M_t^f+N_t^f \\ &= M_t^f+\int_0^t L^{\bar{Q}^{u,n}}f(X_s)ds - <M^f,M>_t \\ &\quad -\int_0^t f(X_s)d(A_s^{-u^*}-N_s^{|u|_E}). \end{aligned} \tag{2.14}$$

By Itô's formula (cf. [24, II, Theorem 33]), (2.14) and (2.4)-(2.6), we obtain that for $t<\tau_{E_n}$

$$\begin{aligned} & f(X_t)\ e^{M_t^{-u^*}-N_t^{|u|_E}} \\ &= f(X_0)+\int_0^t e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}}df(X_s)+\int_0^t e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}}f(X_{s-})d(M_s^{-u^*}-N_s^{|u|_E}) \\ &\quad +\frac{1}{2}\int_0^t e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}}f(X_{s-})d<M^{-u^*,c}>_s+\int_0^t e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}}d<M^{f,c},M^{-u^*,c}>_s \\ &\quad +\sum_{s\leq t}[f(X_s)e^{M_s^{-u^*}-N_s^{|u|_E}}-f(X_{s-})e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}} \\ &\qquad -e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}}\triangle f(X_s)-f(X_{s-})e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}}\triangle M_s^{-u^*}] \end{aligned}$$

$$
\begin{aligned}
&= f(X_0)+\int_0^t e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}}dM_s^f+\int_0^t e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}}L^{\bar{Q}^{u,n}}f(X_s)ds\\
&\quad -\int_0^t e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}}d<M^f,M>_s-\int_0^t e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}}f(X_s)d(A_s^{-u^*}-N_s^{|u|_E})\\
&\quad +\int_0^t e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}}f(X_{s-})d(M_s^{-u^*}-N_s^{|u|_E})\\
&\quad +\frac{1}{2}\int_0^t e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}}f(X_{s-})d<M^{-u^*,c}>_s+\int_0^t e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}}d<M^{f,c},M^{-u^*,c}>_s\\
&\quad +\sum_{s\le t}[f(X_s)e^{M_s^{-u^*}-N_s^{|u|_E}}-f(X_{s-})e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}}\\
&\qquad -e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}}\triangle f(X_s)-f(X_{s-})e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}}\triangle M_s^{-u^*}]\\
&= \Big\{f(X_0)+\int_0^t e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}}dM_s^f+\int_0^t e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}}L^{\bar{Q}^{u,n}}f(X_s)ds\\
&\qquad +\int_0^t e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}}f(X_{s-})dM_s^{-u^*}\Big\}\\
&\quad +\Big\{-\int_0^t e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}}f(X_s)dA_s^{-u^*}+\frac{1}{2}\int_0^t e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}}f(X_{s-})d<M^{-u^*,c}>_s\\
&\quad -\int_0^t e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}}d<M^f,M>_s+\int_0^t e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}}d<M^{f,c},M^{-u^*,c}>_s\\
&\quad +\sum_{s\le t}[f(X_s)e^{M_s^{-u^*}-N_s^{|u|_E}}-f(X_{s-})e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}}\\
&\qquad -e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}}\triangle f(X_s)-f(X_{s-})e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}}\triangle M_s^{-u^*}]\Big\}\\
&= \Big\{f(X_0)+\int_0^t e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}}dM_s^f+\int_0^t e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}}L^{\bar{Q}^{u,n}}f(X_s)ds\\
&\qquad +\int_0^t e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}}f(X_{s-})dM_s^{-u^*}\Big\}\\
&\quad +\Big\{-\int_0^t e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}}f(X_{s-})dB_s^p-\int_0^t e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}}d<M^{f,d},M^d>_s\\
&\quad -\int_0^t e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}}d<M^{f,d},M^{-u^*,d}>_s\\
&\quad +\sum_{s\le t}[f(X_s)e^{M_s^{-u^*}-N_s^{|u|_E}}-f(X_{s-})e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}}\\
&\qquad -e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}}\triangle f(X_s)-f(X_{s-})e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}}\triangle M_s^{-u^*}]\Big\}\\
&:= I+II. \qquad (2.15)
\end{aligned}
$$

Note that

$$
II = -\int_0^t e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}}f(X_{s-})dB_s^p+\sum_{s\le t}[-e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}}\triangle f(X_s)\triangle B_s
$$

$$
\begin{aligned}
& -e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}}\triangle f(X_s)\triangle M_s^{-u^*}]+\sum_{s\le t}[f(X_s)e^{M_s^{-u}-N_s^{|u|_E}} \\
& -f(X_{s-})e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}}-e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}}\triangle f(X_s)-f(X_{s-})e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}}\triangle M_s^{-u^*}] \\
= & -\int_0^t e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}}f(X_{s-})dB_s^p+\sum_{s\le t}[-e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}}\triangle f(X_s)(e^{\triangle M_s^{-u^*}}-1) \\
& +f(X_s)e^{M_s^{-u^*}-N_s^{|u|_E}}-f(X_{s-})e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}}-e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}}\triangle f(X_s)) \\
& -f(X_{s-})e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}}\triangle M_s^{-u^*}] \\
= & -\int_0^t e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}}f(X_{s-})dB_s^p+\sum_{s\le t}[e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}}f(X_{s-})e^{\triangle M_s^{-u^*}} \\
& -f(X_{s-})e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}}-f(X_{s-})e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}}\triangle M_s^{-u^*}] \\
= & -\int_0^t e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}}f(X_{s-})dB_s^p+\int_0^t e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}}f(X_{s-})dB_s \\
= & \int_0^t e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}}f(X_{s-})dM_s^d. \qquad (2.16)
\end{aligned}
$$

Therefore (2.12) follows from (2.15) and (2.16). □

2.2 The bilinear form associated with $(\bar{P}_t^{u,n})_{t\ge0}$ on $L^2(E_n;e^{-2u^*}m)$

For $n\in\mathbf{N}$, since $u^*\cdot I_{E_n}$ is bounded, $(\bar{P}_t^{u,n})_{t\ge0}$ is also a strongly continuous semigroup on $L^2(E_n;e^{-2u^*}m)$ by Theorem 2.1. In this subsection, we will study the bilinear form associated with $(\bar{P}_t^{u,n})_{t\ge0}$ on $L^2(E_n;e^{-2u^*}m)$.

Define $D(\mathcal{E})_{n,b}:=D(\mathcal{E})_n\cap L^\infty(E;m)$. Let $f,g\in D(\mathcal{E})_{n,b}$. Note that $e^{-2u^*}g=(e^{-2u^*}-1)g+g\in D(\mathcal{E})_{n,b}$. Define

$$\mathcal{E}^{u,n}(f,g):=\bar{Q}^{u,n}(f,e^{-2u^*}g),\quad f,g\in D(\mathcal{E})_{n,b}. \qquad (2.17)$$

Then, by Theorem 2.1, we get

$$\mathcal{E}^{u,n}(f,g)=\lim_{t\to0}\frac{1}{t}(f-\bar{P}_t^{u,n}f,e^{-2u^*}g)_m=\lim_{t\to0}\frac{1}{t}(f-\bar{P}_t^{u.n}f,g)_{e^{-2u^*}m}. \qquad (2.18)$$

$(\mathcal{E}^{u,n},D(\mathcal{E})_{n,b})$ is called the bilinear from associated with $(\bar{P}_t^{u,n})_{t\ge0}$ on $L^2(E_n;e^{-2u^*}m)$.

Note that

$$
\begin{aligned}
& <M^f,M^d>_t \\
= & [M^f,M^d]_t^p \\
= & \left\{\sum_{s\le t}[f(X_s)-f(X_{s-})][e^{(u^*(X_{s-})-u^*(X_s))}-1-(u^*(X_{s-})-u^*(X_s))]\right\}^p
\end{aligned}
$$

$$= \int_0^t \int_{E_\triangle} [f(y) - f(X_s)][e^{(u^*(X_s)-u^*(y))} - 1 - (u^*(X_s) - u^*(y))]N(X_s, dy)dH_s.$$

Then

$$\int_E g d\mu_{<M^f,M^d>}$$
$$= \int_E \int_{E_\triangle} g(x)[f(y) - f(x)][e^{(u^*(x)-u^*(y))} - 1 - (u^*(x) - u^*(y))]N(x, dy)\nu(dx). \tag{2.19}$$

By (2.7) and (2.8), we get

$$\begin{aligned}\int_E fg d\mu_{-u} &= \int_E \int_{E_\triangle} f(x)g(x)[e^{(u^*(x)-u^*(y))} - 1 - (u^*(x) - u^*(y))]N(x, dy)\nu(dx) \\ &\quad + \frac{1}{2}\int_E fg d\mu_{<M^{-u^*,c}>} + \int_E fg d\eta_u - \int_E fg|u|_E dm. \end{aligned} \tag{2.20}$$

Similar to [17, Theorem 5.3.1] (cf. also [23, Chapter 5]), we can show that $J(dx, dy) = \frac{1}{2}N(y, dx)\nu(dy)$ and $K(dx) = N(x, \triangle)\nu(dx)$. Therefore, we obtain by (2.17), (2.9), (2.19) and (2.20) that

$$\begin{aligned}\mathcal{E}^{u,n}(f, g) &= \bar{Q}^{u.n}(f, e^{-2u^*}g) \\ &= \mathcal{E}(f, e^{-2u^*}g) - \int_E e^{-2u^*} g d\mu_{<M^f,M>} - \int_E e^{-2u^*} fg d\mu_{-u} \\ &= \mathcal{E}(f, e^{-2u^*}g) - \int_E e^{-2u^*} g d\mu_{<M^f,M^{-u^*}>} - \int_E e^{-2u^*} g d\mu_{<M^f,M^d>} \\ &\quad - \int_E e^{-2u^*} fg d\mu_{-u} \\ &= \mathcal{E}(f, e^{-2u^*}g) - \int_E e^{-2u^*} g d\mu_{<M^f,M^{-u^*}>} \\ &\quad -2\int_{E\times E-d} e^{-2u^*(y)} g(y)f(x)[e^{(u^*(y)-u^*(x))} - 1 - (u^*(y) - u^*(x))]J(dx, dy) \\ &\quad -\frac{1}{2}\int_E e^{-2u^*} fg d\mu_{<M^{-u^*,c}>} - \mathcal{E}(|u|_E, e^{-2u^*} fg). \end{aligned} \tag{2.21}$$

Theorem 2.3. *For each $n \in \mathbf{N}$, under either the assumption of Theorem 1.1 or Theorem 1.2, we have*

$$\mathcal{E}^{u,n}(f, g) = Q^u(fe^{-u^*}, ge^{-u^*}), \quad f, g \in D(\mathcal{E})_{n,b}. \tag{2.22}$$

Proof. We fix an $n \in \mathbf{N}$. Define

$$\Psi^{u^*,n}(f, g) := \mathcal{E}(f, e^{-2u^*}g) - \int_E e^{-2u^*} g d\mu_{<M^f,M^{-u^*}>}$$

$$\begin{aligned}&-2\int_{E\times E-d}e^{-2u^*(y)}g(y)f(x)[e^{(u^*(y)-u^*(x))}-1-(u^*(y)-u^*(x))]J(dx,dy)\\&-\frac{1}{2}\int_E e^{-2u^*}fgd\mu_{<M^{-u^*,c}>},\quad f,g\in D(\mathcal{E})_{n,b}.\end{aligned}\tag{2.23}$$

Then, by (2.21) and (1.1), we find that (2.22) is equivalent to

$$\Psi^{u^*,n}(f,g)=\mathcal{E}(fe^{-u^*},ge^{-u^*})+\mathcal{E}(u^*,e^{-2u^*}fg),\quad f,g\in D(\mathcal{E})_{n,b}.\tag{2.24}$$

Since $u^*\cdot I_{E_n}$ is bounded, there exists $l_0\in\mathbf{N}$ such that $|u^*(x)|\le l_0$ for all $x\in E_n$. For $l\in\mathbf{N}$, define $u_l^*:=((-l)\vee u^*)\wedge l$. Then $u_l^*\in D(\mathcal{E})_b$ and $u^*=u_l^*$ on E_n for $l\ge l_0$. Similar to [17, Lemma 5.3.1], we can show that $\mu_{<M^{-u^*,c}>}|_{E_n}=\mu_{<M^{-u_l^*,c}>}|_{E_n}$ for $l\ge l_0$. For $\phi\in D(\mathcal{E})_b$, we define

$$\begin{aligned}\Psi^{\phi,n}(f,g):=\mathcal{E}(f,e^{-2\phi}g)-\int_E e^{-2\phi}gd\mu_{<M^f,M^{-\phi}>}&\\-2\int_{E\times E-d}e^{-2\phi(y)}g(y)f(x)[e^{(\phi(y)-\phi(x))}-1-(\phi(y)-\phi(x))]J(dx,dy)&\\-\frac{1}{2}\int_E e^{-2\phi}fgd\mu_{<M^{-\phi,c}>},\quad f,g\in D(\mathcal{E})_{n,b}.&\end{aligned}\tag{2.25}$$

Then, by (2.23) and (2.25), we find that for $l\ge l_0$

$$\Psi^{u^*,n}(f,g)=\Psi^{u_l^*,n}(f,g)+\int_E e^{-2u^*}gd\mu_{<M^f,M^{u^*-u_l^*}>},\quad f,g\in D(\mathcal{E})_{n,b}.$$

Note that by [23, (5.1.3)]

$$\begin{aligned}\left|\int_E e^{-2u^*}gd\mu_{<M^f,M^{u^*-u_l^*}>}\right| &\le 2e^{2l_0}\|g\|_\infty\mathcal{E}(f,f)^{\frac{1}{2}}\mathcal{E}(u^*-u_l^*,u^*-u_l^*)^{\frac{1}{2}}\\&\to 0\ \text{ as } l\to\infty,\end{aligned}$$

and

$$\mathcal{E}(u_l^*,e^{-2u^*}fg)\to\mathcal{E}(u^*,e^{-2u^*}fg)\ \text{ as } l\to\infty.$$

Hence, to establish (2.24), it is sufficient to show that for any $\phi\in D(\mathcal{E})_b$ and $f,g\in D(\mathcal{E})_{n,b}$

$$\Psi^{\phi,n}(f,g)=\mathcal{E}(fe^{-\phi},ge^{-\phi})+\mathcal{E}(\phi,e^{-2\phi}fg).\tag{2.26}$$

Let $\phi\in D(\mathcal{E})_b$. By [23, (5.3.2)], we have

$$\int gd\mu_{<M^f,M^{-\phi}>}=-\mathcal{E}(f,g\phi)-\mathcal{E}(\phi,gf)+\mathcal{E}(f\phi,g).\tag{2.27}$$

By (2.25) and (2.27), we find that (2.26) is equivalent to

$$\begin{aligned}&\mathcal{E}(f,e^{-2\phi}g)+\mathcal{E}(f,e^{-2\phi}g\phi)-\mathcal{E}(f\phi,e^{-2\phi}g)\\&\quad-2\int_{E\times E-d}e^{-2\phi(y)}g(y)f(x)[e^{(\phi(y)-\phi(x))}-1-(\phi(y)-\phi(x))]J(dx,dy)\\&\quad-\frac{1}{2}\int_E e^{-2\phi}fgd\mu_{<M^{-\phi,c}>}\\&=\mathcal{E}(fe^{-\phi},ge^{-\phi}).\end{aligned}\tag{2.28}$$

Denote by $M_t^{-\phi,j}$ and $M_t^{-\phi,k}$ the jumping and killing parts of $M_t^{-\phi}$, respectively. Then, similar to [17, (5.3.9) and (5.3.10)], we get

$$\mu_{<M^{-\phi,j}>}(dx) = 2\int_E (\phi(x)-\phi(y))^2 J(dy,dx) \ \text{ and } \ \mu_{<M^{-\phi,k}>}(dx) = \phi^2(x)K(dx).$$

Thus, for any $w \in D(\mathcal{E})_b$, we have

$$\begin{aligned}\int_E w d\mu_{<M^{-\phi,c}>} &= \int_E wd(\mu_{<M^{-\phi}>} - \mu_{<M^{-\phi,j}>} - \mu_{<M^{-\phi,k}>}) \\ &= 2\mathcal{E}(\phi,\phi w) - \mathcal{E}(\phi^2, w) \\ &\quad -2\int_{E\times E-d}(\phi(y)-\phi(x))^2 w(y)J(dx,dy) - \int_E w\phi^2 dK.\end{aligned} \tag{2.29}$$

By (2.29), we find that (2.28) is equivalent to

$$\begin{aligned}&\mathcal{E}(f,e^{-2\phi}g) + \mathcal{E}(f,e^{-2\phi}g\phi) - \mathcal{E}(f\phi,e^{-2\phi}g) - \mathcal{E}(\phi,e^{-2\phi}\phi fg) + \frac{1}{2}\mathcal{E}(\phi^2,e^{-2\phi}fg) \\ &\quad -2\int_{E\times E-d} e^{-2\phi(y)}g(y)f(x)[e^{(\phi(y)-\phi(x))} - 1 - (\phi(y)-\phi(x))]J(dx,dy) \\ &\quad +\int_{E\times E-d}(\phi(y)-\phi(x))^2 e^{-2\phi(y)}f(y)g(y)J(dx,dy) + \frac{1}{2}\int_E e^{-2\phi}fg\phi^2 dK \\ &= \mathcal{E}(fe^{-\phi}, ge^{-\phi}).\end{aligned} \tag{2.30}$$

Proof of (2.30) under the assumption of Theorem 1.1.

Denote by $\tilde{\mathcal{E}}$ the symmetric part of $\mathcal{E}$. Then $(\tilde{\mathcal{E}}, D(\mathcal{E}))$ is a regular symmetric Dirichlet form. Denote by $\tilde{J}$ and $\tilde{K}$ the jumping and killing measures of $(\tilde{\mathcal{E}}, D(\mathcal{E}))$, respectively. Then

$$\begin{aligned}&\int_{E\times E-d}(\phi(y)-\phi(x))^2 J(dx,dy) + \int_E \phi^2 dK \\ &\quad \le 2\left\{\int_{E\times E-d}(\phi(y)-\phi(x))^2\tilde{J}(dx,dy) + \int_E \phi^2 d\tilde{K}\right\} \\ &\quad \le 2\mathcal{E}(\phi,\phi)\end{aligned} \tag{2.31}$$

and

$$\begin{aligned}&\int_{E\times E-d}[e^{(\phi(y)-\phi(x))} - 1 - (\phi(y)-\phi(x))]J(dx,dy) \\ &\quad \le C_{\|\phi\|_\infty}\int_{E\times E-d}(\phi(y)-\phi(x))^2 J(dx,dy) \\ &\quad \le C_{\|\phi\|_\infty}\mathcal{E}(\phi,\phi)\end{aligned} \tag{2.32}$$

for some constant $C_{\|\phi\|_\infty} > 0$. Hence, to establish (2.30) for $\phi \in D(\mathcal{E})_b$ and $f,g \in D(\mathcal{E})_{n,b}$, it is sufficient to establish (2.30) for $\phi, f, g \in D := C_0(E)\cap D(\mathcal{E})$ by virtue of the density of D in $D(\mathcal{E})$ and approximation.

By [20, Theorem 4.8 and Proposition 5.1], we have the following Beurling-Deny decomposition

$$\begin{aligned}\mathcal{E}(f,g) &= \mathcal{E}^c(f,g) + SPV\int_{E\times E-d} 2(f(y)-f(x))g(y)J(dx,dy)\\ &\quad + \int_E fgdK, \quad f,g\in D(\mathcal{E})_b, \qquad (2.33)\end{aligned}$$

where $SPV\int$ denotes the symmetric principle value integral (see [20, Definition 2.5]) and $\mathcal{E}^c(f,g)$ satisfies the left strong local property in the sense that $\mathcal{E}^c(f,g)=0$ if f is constant $\mathcal{E}$-q.e. on a quasi-open set containing the quasi-support of g (see [20, Theorem 4.1]). By (2.33), we obtain that for any $w\in D(\mathcal{E})_b$,

$$\begin{aligned}&2\mathcal{E}(\phi,\phi w) - \mathcal{E}(\phi^2,w)\\ &\qquad -2\int_{E\times E-d}(\phi(y)-\phi(x))^2 w(y)J(dx,dy) - \int_E w\phi^2 dK\\ &= 2\mathcal{E}^c(\phi,\phi w) - \mathcal{E}^c(\phi^2,w).\end{aligned}$$

Hence (2.30) is equivalent to

$$\begin{aligned}&\mathcal{E}(f,e^{-2\phi}g) + \mathcal{E}(f,e^{-2\phi}g\phi) - \mathcal{E}(f\phi,e^{-2\phi}g) - \mathcal{E}^c(\phi,e^{-2\phi}\phi fg) + \frac{1}{2}\mathcal{E}^c(\phi^2,e^{-2\phi}fg)\\ &\qquad -2\int_{E\times E-d} e^{-2\phi(y)}g(y)f(x)[e^{(\phi(y)-\phi(x))}-1-(\phi(y)-\phi(x))]J(dx,dy)\\ &= \mathcal{E}(fe^{-\phi},ge^{-\phi}). \qquad (2.34)\end{aligned}$$

In the following, we will establish (2.34) by showing that its left hand side and its right hand side have the same diffusion, jumping and killing parts. We assume without loss of generality that $\phi,f,g\in D$.

First, let us consider the diffusion parts of both sides of (2.34). Following [21, (3.4)], we introduce a linear functional $< L(w,v),\cdot >$ for $w,v\in D$ by

$$< L(w,v), f > := \check{\mathcal{E}}^c(w,vf) := \frac{1}{2}(\mathcal{E}^c(w,vf) - \hat{\mathcal{E}}^c(w,vf)), \quad f\in D, \qquad (2.35)$$

where $\hat{\mathcal{E}}^c$ is the left strong local part of the dual Dirichlet form $(\hat{\mathcal{E}},D(\mathcal{E}))$. Define

$$\begin{aligned}D_{\mathrm{loc}} := \{w|\ &\text{for any relatively compact open set } G \text{ of } E, \text{there}\\ &\text{exists a function } v\in D \text{ such that } w=v \text{ on } G\}.\end{aligned}$$

Then, the linear functional $< L(w,v),\cdot >$ can be extended and defined for any $w,v\in D_{\mathrm{loc}}$ (cf. [21, Definition 3.6]). Note that J_1 is assumed to be finite. Similar to [21, Theorem 3.8], we can prove the following lemma.

Lemma 2.4. *Let $w_1,\dots,w_l,v\in D_{\mathrm{loc}}$ and $f\in D$. Denote $w:=(w_1,\dots,w_l)$. If $\psi\in C^2(\mathbf{R}^l)$, then $\psi(w)\in D_{\mathrm{loc}}$, $\psi_{x_i}(w)\in D_{\mathrm{loc}}$, $1\le i\le l$, and*

$$< L(\psi(w),v), f > = \sum_{i=1}^{l} < L(w_i,v), \psi_{x_i}(w)f > . \qquad (2.36)$$

By (2.35) and (2.36), we get

$$
\begin{aligned}
&\check{\mathcal{E}}^c(f, e^{-2\phi}g) + \check{\mathcal{E}}^c(f, e^{-2\phi}g\phi) - \check{\mathcal{E}}^c(f\phi, e^{-2\phi}g) \\
&\qquad -\check{\mathcal{E}}^c(\phi, e^{-2\phi}\phi fg) + \frac{1}{2}\check{\mathcal{E}}^c(\phi^2, e^{-2\phi}fg) \\
&= \check{\mathcal{E}}^c(f, e^{-2\phi}g) + \check{\mathcal{E}}^c(f, e^{-2\phi}g\phi) - \check{\mathcal{E}}^c(f\phi, e^{-2\phi}g) \\
&= \check{\mathcal{E}}^c(f, e^{-2\phi}g) - \check{\mathcal{E}}^c(\phi, e^{-2\phi}fg) \\
&= \check{\mathcal{E}}^c(f, e^{-2\phi}g) + \check{\mathcal{E}}^c(e^{-\phi}, e^{-\phi}fg) \\
&= \check{\mathcal{E}}^c(fe^{-\phi}, ge^{-\phi}).
\end{aligned} \tag{2.37}
$$

By LeJan's formula (cf. [17, Theorem 3.2.2 and Page 117], we can check that

$$
\begin{aligned}
&\tilde{\mathcal{E}}^c(f, e^{-2\phi}g) + \tilde{\mathcal{E}}^c(f, e^{-2\phi}g\phi) - \tilde{\mathcal{E}}^c(f\phi, e^{-2\phi}g) \\
&\qquad -\tilde{\mathcal{E}}^c(\phi, e^{-2\phi}\phi fg) + \frac{1}{2}\tilde{\mathcal{E}}^c(\phi^2, e^{-2\phi}fg) \\
&= \frac{1}{2}\int_E d\tilde{\mu}^c_{<f, e^{-2\phi}g>} + \frac{1}{2}\int_E d\tilde{\mu}^c_{<f, e^{-2\phi}g\phi>} - \frac{1}{2}\int_E d\tilde{\mu}^c_{<f\phi, e^{-2\phi}g>} \\
&\qquad -\frac{1}{2}\int_E d\tilde{\mu}^c_{<\phi, e^{-2\phi}\phi fg>} + \frac{1}{4}\int_E d\tilde{\mu}^c_{<\phi^2, e^{-2\phi}fg>} \\
&= \frac{1}{2}\int_E d\tilde{\mu}^c_{<fe^{-\phi}, ge^{-\phi}>} \\
&= \tilde{\mathcal{E}}^c(fe^{-\phi}, ge^{-\phi}),
\end{aligned} \tag{2.38}
$$

where $\tilde{\mathcal{E}}^c$ denotes the strong local part of $(\tilde{\mathcal{E}}, D(\mathcal{E}))$ and $\tilde{\mu}^c$ denotes the local part of energy measure w.r.t. $(\tilde{\mathcal{E}}, D(\mathcal{E}))$. Then the diffusion parts of both sides of (2.34) are equal by (2.37) and (2.38).

For the jumping parts of (2.34), we have

$$
\begin{aligned}
&\mathcal{E}^j(f, e^{-2\phi}g) + \mathcal{E}^j(f, e^{-2\phi}g\phi) - \mathcal{E}^j(f\phi, e^{-2\phi}g) - \mathcal{E}^j(fe^{-\phi}, ge^{-\phi}) \\
&\qquad -2\int_{E\times E-d} e^{-2\phi(y)}g(y)f(x)[e^{(\phi(y)-\phi(x))} - 1 - (\phi(y)-\phi(x))]J(dx,dy) \\
&= 2SPV\int_{E\times E-d}\{(f(y)-f(x))e^{-2\phi(y)}g(y) + (f(y)-f(x))\phi(y)e^{-2\phi(y)}g(y) \\
&\qquad -(f(y)\phi(y)-f(x)\phi(x))e^{-2\phi(y)}g(y) - (f(y)e^{-\phi(y)} - f(x)e^{-\phi(x)})e^{-\phi(y)}g(y) \\
&\qquad -e^{-2\phi(y)}g(y)f(x)[e^{(\phi(y)-\phi(x))} - 1 - (\phi(y)-\phi(x))]\}J(dx,dy) \\
&= 0.
\end{aligned}
$$

For the killing parts of (2.34), we have

$$
\begin{aligned}
&\mathcal{E}^k(f, e^{-2\phi}g) + \mathcal{E}^k(f, e^{-2\phi}g\phi) - \mathcal{E}^k(f\phi, e^{-2\phi}g) - \mathcal{E}^k(fe^{-\phi}, ge^{-\phi}) \\
&= \int_E (fe^{-2\phi}g + fe^{-2\phi}g\phi - f\phi e^{-2\phi}g - fe^{-2\phi}g)dK \\
&= 0.
\end{aligned}
$$

The proof is complete.

Proof of (2.30) under the assumption of Theorem 1.2.

Let G be a relatively compact open subset of U such that the distance between the boundary of G and that of U is greater than some constant $\delta > 0$. Then, similar to [21, Theorem 4.8], we can show that $(\mathcal{E}, C_0^\infty(G))$ has the following representation:

$$
\begin{aligned}
\mathcal{E}(w,v) \;=\;& \sum_{i,j=1}^d \int_U \frac{\partial w}{\partial x_i}\frac{\partial v}{\partial x_j} d\nu_{ij}^G + \sum_{i=1}^d < F_i^G, \frac{\partial w}{\partial x_i} v > \\
&+ SPV \int_{U\times U-d} 2\left(\sum_{i=1}^d (y_i - x_i)\frac{\partial w}{\partial y_i}(y) I_{\{|x-y|\le\frac{\delta}{2}\}}(x,y)\right) v(y)\tilde{J}(dx,dy) \\
&+ \int_{U\times U-d} 2\left(w(y) - w(x) - \sum_{i=1}^d (y_i - x_i)\frac{\partial w}{\partial y_i}(y) I_{\{|x-y|\le\frac{\delta}{2}\}}(x,y)\right) v(y) J(dx,dy) \\
&+ \int_U wv dK, \qquad w, v \in C_0^\infty(G),
\end{aligned}
\tag{2.39}
$$

where $\{\nu_{ij}^G\}_{1\le i,j\le d}$ are signed Radon measures on U such that for every $K \subset U$, K is compact, $\nu_{ij}^G(K) = \nu_{ji}^G(K)$ and $\sum_{i,j=1}^d \xi_i\xi_j\nu_{ij}^G(K) \ge 0$ for all $\xi = (\xi_1, \ldots, \xi_d) \in \mathbf{R}^d$, $\{F_i^G\}_{1\le i\le d}$ are generalized functions on U.

By (2.39), we can check that (2.30) holds for all $\phi, f, g \in C_0^\infty(U)$. Therefore (2.30) holds for $\phi \in D(\mathcal{E})_b$ and $f, g \in D(\mathcal{E})_{n,b}$ by (2.31), (2.32) and approximation. The proof is complete. □

2.3 Proof of Theorems 1.1 and 1.2 and some remarks

Proof. By Theorem 2.1, for each $n \in \mathbf{N}$, $(\bar{P}_t^{u,n})_{t\ge0}$ is a strongly continuous semigroup of bounded operators on $L^2(E_n; m)$ with $\|\bar{P}_t^{u,n}\|_2 \le e^{\beta_n t}$ for every $t > 0$ and some constant $\beta_n > 0$. Moreover, the coercive closed form associated with $(e^{-\beta_n t}\bar{P}_t^{u,n})_{t\ge0}$ is given by $(\bar{Q}_{\beta_n}^{u,n}, D(\mathcal{E})_n)$. Note that $(\bar{P}_t^{u,n})_{t\ge0}$ is also a strongly continuous semigroup of bounded operators on $L^2(E_n; e^{-2u^*}m)$ and the bilinear from associated with $(\bar{P}_t^{u,n})_{t\ge0}$ on $L^2(E_n; e^{-2u^*}m)$ is given by $(\mathcal{E}^{u,n}, D(\mathcal{E})_{n,b})$ (see (2.18)).

Define

$$P_t^{u,n} f(x) := E_x[e^{N_t^u} f(X_t); t < \tau_{E_n}].$$

Then

$$
\begin{aligned}
P_t^{u,n} f(x) &= E_x[e^{N_t^{u^*} - N_t^{|u|_E}} f(X_t); t < \tau_{E_n}] \\
&= E_x[e^{u^*(X_t) - u^*(X_0) + M_t^{-u^*} - N_t^{|u|_E}} f(X_t); t < \tau_{E_n}] \\
&= e^{-u^*(x)} \bar{P}_t^{u,n}(e^{u^*} f)(x).
\end{aligned}
\tag{2.40}
$$

Hence $(P_t^{u,n})_{t\geq 0}$ is a strongly continuous semigroup of bounded operators on $L^2(E_n;m)$. Let $(Q^{u,n}, D(\mathcal{E})_{n,b})$ be the restriction of Q^u to $D(\mathcal{E})_{n,b}$. Then, by (2.40), (2.18) and Theorem 2.3, we know that the bilinear from associated with $(P_t^{u,n})_{t\geq 0}$ on $L^2(E_n;m)$ is given by $(Q^{u,n}, D(\mathcal{E})_{n,b})$. That is,

$$Q^{u,n}(f,g) = \lim_{t\to 0}\frac{1}{t}(f - P_t^{u,n}f, g)_m, \quad f,g\in D(\mathcal{E})_{n,b}. \tag{2.41}$$

(i) Suppose that there exists a constant $\alpha_0 \geq 0$ such that

$$Q^u(f,f) \geq -\alpha_0(f,f)_m, \quad \forall f\in D(\mathcal{E})_b.$$

For $n\in\mathbf{N}$, let $(L^n, D(L^n))$ be the generator of $(P_t^{u,n})_{t\geq 0}$ on $L^2(E_n;m)$. Then $D(L^n-\alpha_0)$ is dense in $L^2(E_n;m)$.

Define

$$\bar{L}^n f(x) = e^{u^*(x)}L^n(e^{-u^*}f)(x), \quad f\in D(\bar{L}^n) := \{e^{u^*}g | g\in D(L^n)\}. \tag{2.42}$$

Then, by (2.40), $(\bar{L}^n, D(\bar{L}^n))$ is the generator of $(\bar{P}_t^{u,n})_{t\geq 0}$ on $L^2(E_n; e^{-2u^*}m)$. $(\bar{L}^n, D(\bar{L}^n))$ is also the generator of $(\bar{P}_t^{u,n})_{t\geq 0}$ on $L^2(E_n;m)$ due to the boundedness of u^* on E_n. Since $(e^{-\beta_n t}\bar{P}_t^{u,n})_{t\geq 0}$ is a strongly continuous contraction semigroup on $L^2(E_n;m)$, Range$(\lambda - \bar{L}^n) = L^2(E_n;m)$ for all $\lambda > \beta_n$. Hence Range$(\lambda - (L^n-\alpha_0)) = L^2(E_n;m)$ for all $\lambda > \beta_n - \alpha_0$ by (2.42).

Let $f\in L^2(E_n;m)$. Then, for any $\alpha > 0$, we obtain by (2.41) that

$$\begin{aligned}
\|[\alpha - (L^n-\alpha_0)]f\|_2\cdot\|f\|_2 &= \|[(\alpha+\alpha_0) - L^n]f\|_2\cdot\|f\|_2\\
&\geq ([(\alpha+\alpha_0)-L^n]f,f)_m\\
&= Q^{u,n}(f,f) + (\alpha+\alpha_0)(f,f)_m\\
&\geq \alpha(f,f)_m.
\end{aligned}$$

Hence $L^n-\alpha_0$ is dissipative on $L^2(E_n;m)$. Therefore $(e^{-\alpha_0 t}P_t^{u,n})_{t\geq 0}$ is a strongly continuous contraction semigroup on $L^2(E_n;m)$ by the Hille-Yosida theorem (cf. [14, Chapter 1, Theorem 2.6]).

Let $g\in L^2(E;m)$ and $t>0$. Then

$$\begin{aligned}
\|P_t^u g\|_2 &\leq \|P_t^u|g|\,\|_2\\
&= \lim_{l\to\infty}\|P_t^u|g\cdot I_{E_l}|\,\|_2\\
&\leq \liminf_{l\to\infty}\liminf_{n\to\infty}\|P_t^{u,n}|g\cdot I_{E_l}|\,\|_2\\
&\leq e^{\alpha_0 t}\|g\|_2.
\end{aligned}$$

Since $g\in L^2(E;m)$ is arbitrary, we get

$$\|P_t^u\|_2 \leq e^{\alpha_0 t}, \quad \forall t>0.$$

(ii) Suppose that there exists a constant $\alpha_0 \geq 0$ such that

$$\|P_t^u\|_2 \leq e^{\alpha_0 t}, \quad \forall t > 0. \tag{2.43}$$

Let $n \in \mathbf{N}$ and $f \in L^2(E_n; m)$. Then

$$\|P_t^{u,n} f\|_2 \leq \|P_t^{u,n}|f|\,\|_2 \leq \|P_t^u|f|\,\|_2 \leq e^{\alpha_0 t}\|f\|_2.$$

Hence $(e^{-\alpha_0 t} P_t^{u,n})_{t\geq 0}$ is a strongly continuous contraction semigroup on $L^2(E_n; m)$. By (2.41), we get

$$Q^{u,n}(f,f) + \alpha_0(f,f)_m = \lim_{t\to 0} \frac{1}{t}(f - e^{-\alpha_0 t} P_t^{u,n} f, f)_m \geq 0, \quad \forall f \in D(\mathcal{E})_{n,b}. \tag{2.44}$$

By (2.44) and approximation, we find that

$$Q^u(f,f) \geq -\alpha_0(f,f)_m, \quad \forall f \in D(\mathcal{E})_b.$$

Now we show that $(P_t^u)_{t\geq 0}$ is strongly continuous on $L^2(E; m)$. Let $n \in \mathbf{N}$ and $f \in L^2(E_n; m)$ satisfying $f \geq 0$. Then, we obtain by (2.43) and the strong continuity of $(P_t^{u,n})_{t\geq 0}$ that

$$\begin{aligned}
&\limsup_{t\to 0} \|f - e^{-\alpha_0 t} P_t^u f\|_2^2 \\
&\quad = \limsup_{t\to 0} \{2(f - e^{-\alpha_0 t} P_t^u f, f)_m - [(f,f)_m - \|e^{-\alpha_0 t} P_t^u f\|_2^2]\} \\
&\quad \leq 2\limsup_{t\to 0} (f - e^{-\alpha_0 t} P_t^u f, f)_m \\
&\quad \leq 2\limsup_{t\to 0} (f - e^{-\alpha_0 t} P_t^{u,n} f, f)_m \\
&\quad = 0.
\end{aligned}$$

Since f and n are arbitrary, $(P_t^u)_{t\geq 0}$ is strongly continuous on $L^2(E; m)$ by (2.43). The proof is complete. □

Remark 2.5. *Let $u \in D(\mathcal{E})$. Define*

$$B_t^u = \sum_{s\leq t} \left[e^{(u(X_{s-}) - u(X_s))} - 1 - (u(X_{s-}) - u(X_s)) \right]. \tag{2.45}$$

Note that $(B_t^u)_{t\geq 0}$ may not be locally integrable (cf. [4, Theorem 3.3]). To overcome this difficulty, we introduced the nonnegative function u^* and the locally integrable increasing process $(B_t)_{t\geq 0}$ (see (2.2) and (2.3)). This technique has been used in [4] to show that if X is symmetric and $u \in D(\mathcal{E})_e$, then $(P_t^u)_{t\geq 0}$ is strongly continuous if and only if $(Q^u, D(\mathcal{E})_b)$ is lower semi-bounded. Here and henceforth $D(\mathcal{E})_e$ denotes the extended Dirichlet space of $(\mathcal{E}, D(\mathcal{E}))$.

In fact, if we assume that $(\mathcal{E}, D(\mathcal{E}))$ satisfies the strong sector condition instead of the weak sector condition (cf. [22, Pages 15 and 16] for the definitions), then similar to [4, Page 158] we can introduce a function $|u|_E^g$ for each $u \in D(\mathcal{E})_e$.

Define $u^* := u + |u|_E^g$. Using this defined u^*, similar to the above proof of this section, we can show that Theorems 1.1 and 1.2 hold for all $u \in D(\mathcal{E})_e$.

On the other hand, suppose we still assume that $(\mathcal{E}, D(\mathcal{E}))$ satisfies the weak sector condition and $u \in D(\mathcal{E})_e$. Define

$$F_t^u = \sum_{s \le t} \left[e^{(u(X_{s-}) - u(X_s))} - 1 - (u(X_{s-}) - u(X_s)) \right]^2.$$

If $(F_t^u)_{t \ge 0}$ is locally integrable on $[0, \zeta)$ for q.e. $x \in E$, then we can show that Theorems 1.1 and 1.2 still hold. The proof is similar to the above proof of this section but we directly apply the $(B_t^u)_{t \ge 0}$ defined in (2.45) instead of the $(B_t)_{t \ge 0}$ defined in (2.3). Note that if u is lower semi-bounded or $e^u \in D(\mathcal{E})_e$ (cf. [4, Example 3.4 (iii)]), then $(F_t^u)_{t \ge 0}$ is locally integrable on $[0, \zeta)$ for q.e. $x \in E$.

Remark 2.6. *If $(\mathcal{E}, D(\mathcal{E}))$ is a symmetric Dirichlet form, then the assumption of Theorem 1.1 is automatically satisfied. Note that $(P_t^u)_{t \ge 0}$ is symmetric on $L^2(E; m)$. If $(P_t^u)_{t \ge 0}$ is strongly continuous, then (2.43) holds (cf. [10, Remark 1.6(ii)]). Therefore, the following three assertions are equivalent:*
(i) $(Q^u, D(\mathcal{E})_b)$ is lower semi-bounded.
(ii) There exists a constant $\alpha_0 \ge 0$ such that $\|P_t^u\|_2 \le e^{\alpha_0 t}$ for $t > 0$.
(iii) $(P_t^u)_{t \ge 0}$ is strongly continuous on $L^2(E; m)$.

Remark 2.7. *Denote by S the set of all smooth measures on $(E, \mathcal{B}(E))$. Let $\mu = \mu_1 - \mu_2 \in S - S$, $(A_t^1)_{t \ge 0}$ and $(A_t^2)_{t \ge 0}$ be PCAFs with Revuz measures μ_1 and μ_2, respectively. Define*

$$\bar{P}_t^A f(x) = E_x[e^{A_t^2 - A_t^1} f(X_t)], \quad f \ge 0 \text{ and } t \ge 0,$$

and

$$\begin{cases} \mathcal{E}^\mu(f, g) := \mathcal{E}(f, g) + \int_E fg d\mu, \\ f, g \in D(\mathcal{E}^\mu) := \{w \in D(\mathcal{E}) | w \text{ is } (\mu_1 + \mu_2) - \text{square integrable}\}. \end{cases}$$

Then, by a localization argument similar to that used in the proof of Theorems 1.1 and 1.2 (cf. also [7]), we can show that the following two conditions are equivalent:
(i) There exists a constant $\alpha_0 \ge 0$ such that

$$\mathcal{E}^\mu(f, f) \ge -\alpha_0 (f, f)_m, \quad \forall f \in D(\mathcal{E}^\mu).$$

(ii) There exists a constant $\alpha_0 \ge 0$ such that

$$\|\bar{P}_t^A\|_2 \le e^{\alpha_0 t}, \quad \forall t > 0.$$

Furthermore, if one of these conditions holds, then the semigroup $(\bar{P}_t^A)_{t \ge 0}$ is strongly continuous on $L^2(E; m)$.

This result generalizes the corresponding results of [1] and [3]. Note that, similar to Theorems 1.1 and 1.2, it is not necessary to assume that the bilinear form $(\mathcal{E}^\mu, D(\mathcal{E}^\mu))$ satisfies the sector condition.

3 Examples

Example 3.1. In this example, we study the generalized Feynman-Kac semigroup for the non-symmetric Dirichlet form given in [22, II, 2 d)].

Let $d \geq 3$, U be an open set of $\mathbf{R^d}$ and $m = dx$, the Lebesgue measure on U. Let $a_{ij} \in L^1_{\rm loc}(U;dx)$, $1 \leq i,j \leq d$, $b_i, d_i \in L^d_{\rm loc}(U;dx)$, $d_i - b_i \in L^d(U;dx) \cup L^\infty(U;dx)$, $1 \leq i \leq d$, $c \in L^{d/2}_{\rm loc}(U;dx)$. Define for $f, g \in C_0^\infty(U)$

$$\begin{aligned}\mathcal{E}(f,g) &= \sum_{i,j=1}^d \int_U \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j} a_{ij} dx + \sum_{i=1}^d \int_U f \frac{\partial g}{\partial x_i} d_i dx \\ &\quad + \sum_{i=1}^d \int_U \frac{\partial f}{\partial x_i} g b_i dx + \int_U fgc dx.\end{aligned}$$

Denote $\tilde{a}_{ij} := \frac{1}{2}(a_{ij} + a_{ji})$ and $\check{a}_{ij} := \frac{1}{2}(a_{ij} - a_{ji})$, $1 \leq i,j \leq d$. Suppose that the following conditions hold

(C1) There exists $\gamma \in (0,\infty)$ such that $\sum_{i,j=1}^d \tilde{a}_{ij}\xi_i\xi_j \geq \gamma \sum_{i=1}^d |\xi_i|^2, \quad \forall \xi = (\xi_1, \ldots, \xi_d) \in \mathbf{R^d}$.

(C2) $|\check{a}_{ij}| \leq M \in (0,\infty)$ for $1 \leq i,j \leq d$.

(C3) $cdx - \sum_{i=1}^d \frac{\partial d_i}{\partial x_i} \geq 0$ and $cdx - \sum_{i=1}^d \frac{\partial b_i}{\partial x_i} \geq 0$ (in the sense of Schwartz distributions, i.e., $\int_U (cf + \sum_{i=1}^d d_i \frac{\partial f}{\partial x_i})dx, \int_U (cf + \sum_{i=1}^d b_i \frac{\partial f}{\partial x_i})dx \geq 0$ for all $f \in C_0^\infty(U)$ with $f \geq 0$).

Then $(\mathcal{E}, C_0^\infty(U))$ is closable and its closure $(\mathcal{E}, D(\mathcal{E}))$ is a regular Dirichlet form on $L^2(U;dx)$ (see [22, II, Proposition 2.11]).

Let $u \in C_0^\infty(U)$. Then, for $f \in C_0^\infty(U)$, we have

$$\begin{aligned}Q^u(f,f) &= \mathcal{E}(f,f) + \mathcal{E}(u,f^2) \\ &= \sum_{i,j=1}^d \int_U \frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j} a_{ij} dx + \int_U f^2 \left(c(1+u) + \sum_{i=1}^d \frac{\partial u}{\partial x_i} b_i \right) dx \\ &\quad + \int_U \sum_{i=1}^d \frac{\partial f^2}{\partial x_i} \left(\frac{d_i + b_i}{2} + u d_i + \sum_{j=1}^d \frac{\partial u}{\partial x_j} a_{ji} \right) dx.\end{aligned}$$

Suppose that the following condition holds

(C4) There exists a constant $\alpha_0 \geq 0$ such that

$$\left(\alpha_0 + c(1+u) + \sum_{i=1}^d \frac{\partial u}{\partial x_i} b_i \right) dx - \sum_{i=1}^d \frac{\partial (\frac{d_i + b_i}{2} + u d_i + \sum_{j=1}^d \frac{\partial u}{\partial x_j} a_{ji})}{\partial x_i} \geq 0$$

in the sense of Schwartz distribution.

Then $Q^u(f,f) \geq -\alpha_0(f,f)$ for any $f \in C_0^\infty(U)$ and thus for any $f \in D(\mathcal{E})_b$ by approximation.

Let X be a Hunt process associated with $(\mathcal{E}, D(\mathcal{E}))$ and $(P_t^u)_{t\geq 0}$ be the generalized Feynman-Kac semigroup induced by u. Then, by Theorem 1.1 or Theorem 1.2, $(e^{-\alpha_0 t}P_t^u)_{t\geq 0}$ is a strongly continuous contraction semigroup on $L^2(U;dx)$.

Example 3.2. In this example, we study the generalized Feynman-Kac semigroup for the non-symmetric Dirichlet form given in [22, II, 3 e)].

Let E be a locally convex topological real vector space which is a (topological) Souslin space. Let $m := \mu$ be a finite positive measure on $\mathcal{B}(E)$ such that $\operatorname{supp}\mu = E$. Let E' denote the dual of E and ${}_{E'}\langle , \rangle_E : E' \times E \to \mathbf{R}$ the corresponding dualization. Define

$$\mathcal{F}C_b^\infty := \{f(l_1, \dots, l_m) | m \in \mathbf{N}, f \in C_b^\infty(\mathbf{R^m}), l_1, \dots, l_m \in E^{'}\}.$$

Assume that there exists a separable real Hilbert space $(H, \langle , \rangle_H)$ densely and continuously embedded into E. Identifying H with its dual $H^{'}$ we have that

$$E' \subset H \subset E \text{ densely and continuously,}$$

and ${}_{E'}\langle , \rangle_E$ restricted to $E' \times H$ coincides with $\langle , \rangle_H$. For $f \in \mathcal{F}C_b^\infty$ and $z \in E$, define $\nabla u(z) \in H$ by

$$\langle \nabla u(z), h\rangle_H = \frac{\partial u}{\partial h}(z), \quad h \in H.$$

Let $(\mathcal{E}_\mu, \mathcal{F}C_b^\infty)$, defined by

$$\mathcal{E}_\mu(f,g) = \int_E \langle \nabla f, \nabla g\rangle_H d\mu, \quad f, g \in \mathcal{F}C_b^\infty,$$

be closable on $L^2(E;\mu)$ (cf. [22, II, Proposition 3.8 and Corollary 3.13]). Let $\mathcal{L}_\infty(H)$ denote the set of all bounded linear operators on H with operator norm $\| \ \|$. Suppose $z \to A(z), z \in E$, is a map from E to $\mathcal{L}_\infty(H)$ such that $z \to \langle A(z)h_1, h_2\rangle_H$ is $\mathcal{B}(E)$-measurable for all $h_1, h_2 \in H$. Furthermore, suppose that the following conditions hold

(C1) There exists $\gamma \in (0,\infty)$ such that $\langle A(z)h, h\rangle_H \geq \gamma \|h\|_H^2$ for all $h \in H$.

(C2) $\|\tilde{A}\|_\infty \in L^1(E;\mu)$ and $\|\check{A}\|_\infty \in L^\infty(E;\mu)$, where $\tilde{A} := \frac{1}{2}(A+\hat{A})$, $\check{A} := \frac{1}{2}(A-\hat{A})$ and $\hat{A}(z)$ denotes the adjoint of $A(z), z \in E$.

(C3) Let $c \in L^\infty(E,\mu)$ and $b, d \in L^\infty(E \to H;\mu)$ such that for $u \in \mathcal{F}C_b^\infty$ with $u \geq 0$

$$\int_E (\langle d, \nabla u\rangle_H + cu)d\mu \geq 0, \int_E (\langle b, \nabla u\rangle_H + cu)d\mu \geq 0.$$

Define for $f, g \in \mathcal{F}C_b^\infty$

$$\mathcal{E}(f,g) \ \ = \ \ \int_E \langle A\nabla f, \nabla g\rangle_H d\mu + \int_E f\langle d, \nabla g\rangle_H d\mu$$

$$+\int_E \langle b,\nabla f\rangle_H g d\mu+\int_E fgcd\mu.$$

Then $(\mathcal{E},\mathcal{F}C_b^\infty)$ is closable and its closure $(\mathcal{E},D(\mathcal{E}))$ is a quasi-regular Dirichlet form on $L^2(E,\mu)$ (see by [22, II, 3 e)].

Let $u\in\mathcal{F}C_b^\infty$. Then, for $f\in\mathcal{F}C_b^\infty$, we have

$$\begin{aligned}
Q^u(f,f) &= \mathcal{E}(f,f)+\mathcal{E}(u,f^2)\\
&= \int_E \langle A\nabla f,\nabla f\rangle_H d\mu+\int_E (c(1+u)+\langle b,\nabla u\rangle_H)f^2 dx\\
&\quad +\int_E \left\langle \frac{d+b}{2}+ud+A\nabla u,\nabla f^2\right\rangle_H d\mu.
\end{aligned}$$

Suppose that the following condition holds

(C4) There exists a constant $\alpha_0\geq 0$ such that

$$\int_E \left\{(\alpha_0+c(1+u)+\langle b,\nabla u\rangle_H)f+\left\langle \frac{d+b}{2}+ud+A\nabla u,\nabla f\right\rangle_H\right\}d\mu\geq 0$$

for all $f\in\mathcal{F}C_b^\infty$ with $f\geq 0$.

Then $Q^u(f,f)\geq -\alpha_0(f,f)$ for any $f\in\mathcal{F}C_b^\infty$ and thus for any $f\in D(\mathcal{E})_b$ by approximation.

Let X be a μ-tight special standard diffusion process associated with $(\mathcal{E},D(\mathcal{E}))$ and $(P_t^u)_{t\geq 0}$ be the generalized Feynman-Kac semigroup induced by u. Then, by Theorem 1.1, $(e^{-\alpha_0 t}P_t^u)_{t\geq 0}$ is a strongly continuous contraction semigroup on $L^2(E;\mu)$.

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